Construction of TYM via Field Redefinitions [1]

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Abstract

By constructing a nilpotent extended BRST operator $\bar{s}$ that involves the N=2 global supersymmetry transformations of one chirality, we find exact field redefinitions that allows to construct the Topological Yang Mills Theory from the ordinary Euclidean N=2 Super Yang Mills theory in flat space. We also show that the given field redefinitions yield the Baulieu-Singer formulation of Topological Yang Mills theory when after an instanton inspired truncation of the theory is used.

1 Introduction

Topological Yang-Mills (TYM) theory was first constructed by Witten [1] in 1988 as the twisted version of Euclidean N=2 Super Yang-Mills (SYM) theory in order to study the topological invariants of four-manifolds. Soon after [1], it was shown by Baulieu and Singer that TYM can be fully obtained as a pure gauge fixing term (i.e. as an exact BRST term) [2].

Moreover, N=2 SYM and TYM are also intertwined together when physical calculations are considered. For instance, the instanton calculations of N=2 SYM by using semi-classical approximation [3] and the ones of TYM, where no approximation is needed to perform these calculations [4] give the same result. Since some position independent correlators exist in supersymmetric gauge theories [5], one can interpret this result that a subset of correlators of N=2 SYM coincide with a subset of the observables in TYM [4]. The non-renormalition theorems of N=2 SYM can also be proved by using twisted version [6]. As a consequence, one may conclude that twisting can be thought as a variable redefinition in flat space [7].

Therefore, two questions are in order: First of all, is it also possible to write the action of N=2 SYM as an exact term like the twisted version of the theory and second, if twisting can be thought as really a variable redefinition in flat space, is it possible to find these field redefinitions explicitly?

[1] Talk given at the International Workshop Supersymmetries and Quantum Symmetries (SQS05),Dubna, 27-31 July 2005.

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The answers to both of the above questions are found to be affirmative [8] and the strategy to find these answers is to use the BRST formalism (also called BV or field-antifield formalism [9, 10]) that is extended to include global supersymmetry (SUSY). [11, 12, 13, 14, 15].

2 Extended BRST transformations and N=2 SYM action as an exact term

The off-shell Euclidean N=2 SYM action [16] is given as[3],

$$\begin{aligned} I &= Tr\int d^4x(\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{8}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}-\lambda^i \not{D}\bar{\lambda}_i+MD_\mu D_\mu N- \\ &\quad -\frac{i\sqrt{2}}{2}(\lambda_i[\lambda^i,N]+\bar{\lambda}^i[\bar{\lambda}_i,M])-\frac{1}{2}[M,N]^2+\frac{1}{2}\vec{D}.\vec{D}) \end{aligned} \tag{1}$$

where the (anti-hermitian) gauge field A_μ and the scalar fields M , N are singlets, the Weyl spinors $\lambda_{i\alpha}$ $\bar{\lambda}^i_{\dot{\alpha}}$ are doublets and the auxiliary field $\vec{D}$ is a triplet under the $SU(2)_R$ symmetry group.

Since, the action is translation, gauge and N=2 SUSY invariant one can define an extended BRST symmetry [13]:

$$s=s_0-i\xi^iQ_i-i\bar{\xi}_i\bar{Q}^i-i\eta^\mu\partial_\mu \tag{2}$$

where s_0 is the ordinary BRST transformations, Q_i , $\bar{Q}^i$ are chiral and antichiral parts of N=2 SUSY transformations and $\xi^{i\alpha}$, $\bar{\xi}_{i\dot{\alpha}}$ and η_μ are the constant *commuting* chiral, antichiral SUSY ghosts and constant imaginary anticommuting translation ghost respectively. By choosing s-transformations of the ghosts suitably, the extended BRST operator s becomes nilpotent [13] and one can construct a cohomology problem.

On the other hand, from the definition of s it is still possible to derive another nilpotent operator by using a suitable filtration of global ghosts [17]. We choose this filtration to be

$$\mathcal{N}=\bar{\xi}_{i\dot{\alpha}}\frac{\delta}{\delta\bar{\xi}_{i\dot{\alpha}}}+\eta_\mu\frac{\delta}{\delta\eta_\mu} \quad ; \quad s=\sum s^{(n)} \quad , \quad [\mathcal{N},s^{(n)}]=ns^{(n)}, \tag{3}$$

so that the zeroth order in the above expansion is an operator that includes ordinary BRST and chiral SUSY on the space of the fields of the N=2 vector multiplet $M, N, A_\mu, \lambda_i, \bar{\lambda}^i, \vec{D}$

$$\bar{s}:=s^{(0)}=s_0-i\xi^iQ_i \qquad , \qquad \bar{s}^2=0. \tag{4}$$

The cohomology of s is isomorphic to a subset of the cohomology of the filtered operator $\bar{s}$ [17]. The $\bar{s}$ transformation of the fields are given as,

$$\begin{aligned} \bar{s}A_\mu &= D_\mu c-\xi_i e_\mu\bar{\lambda}^i \quad , \quad \bar{s}M=-[c,M]+i\sqrt{2}\xi^i\lambda_i \quad , \quad \bar{s}N=-[c,N] \\ \bar{s}\lambda_i &= -\{c,\lambda_i\}-e_{\mu\nu}\xi_iF_{\mu\nu}+\xi_i[M,N]+\vec{\tau}_i^{\,j}\xi_j.\vec{D} \quad , \quad \bar{s}\bar{\lambda}^i=-\{c,\bar{\lambda}^i\}+i\sqrt{2}\bar{e}_\mu\xi^iD_\mu N \\ \bar{s}\vec{D} &= -[c,\vec{D}]+\vec{\tau}_i^{\,j}(\xi_je_\mu D_\mu\bar{\lambda}^i+i\sqrt{2}\xi^i[\lambda_j,N]) \\ \bar{s}c &= -\frac{1}{2}\{c,c\}+i\sqrt{2}\xi_i\xi^iN \quad , \quad \bar{s}\eta_\mu=\bar{s}\xi_i=\bar{s}\bar{\xi}_i=0 \end{aligned} \tag{5}$$

[3]Our conventions are explained in Ref.[8] in detail.

Since, the actions of SYM theories can be represented as chiral (or antichiral) multiple supervariations of lower dimensional gauge invariant field polynomials [18], it is straightforward to assume that the action can also be written as an $\bar{s}$ exact term of a gauge invariant field polynomial which is independent of Fadeev-Popov ghost fields[4],

$$I = \bar{s}\Psi. \tag{6}$$

It is clear that Ψ, the so called gauge fermion in BV formalism, has negative ghost number, $Gh(\Psi) = -1$. However, since no fields with negative ghost number has been introduced and since we have chosen the gauge fermion to be free of Fadeev-Popov ghosts, the only way to assign a negative ghost number to Ψ is to choose Ψ to depend on the negative powers of the global SUSY ghosts:

$$\Psi = \frac{1}{\xi_k \xi^k} \xi^i \int d^4x \psi_i \tag{7}$$

where ψ_i^α is a dimension 7/2 fermion that is made from the fields of the N=2 vector multiplet. The most general such gauge fermion that is covariant in its Lorentz, spinor and $SU(2)_R$ indices is easy to find:

$$\Psi_E = \frac{1}{\xi_k \xi^k} Tr \int d^4x (\frac{1}{2}\xi^i \lambda_i [M, N] - \frac{1}{2}\xi^i \vec{\tau}_i^j \lambda_j . \vec{D} - \frac{1}{2}\xi^i e_{\mu\nu} \lambda_i F_{\mu\nu} - \frac{i\sqrt{2}}{2} M \xi^i e_\mu D_\mu \bar{\lambda}_i). \tag{8}$$

The coefficients of the terms in Ψ are fixed in order that the $\bar{s}$ variation of Ψ is free of chiral ghosts:

$$\begin{aligned} I_E &= \bar{s}\Psi_E \\ &= Tr \int d^4x (\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{8}\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} - \lambda^i \not{D} \bar{\lambda}_i + M D_\mu D_\mu N \\ &\qquad - \frac{i\sqrt{2}}{2}(\lambda_i [\lambda^i, N] + \bar{\lambda}^i [\bar{\lambda}_i, M]) - \frac{1}{2}[M, N]^2 + \frac{1}{2}\vec{D}.\vec{D}) \end{aligned} \tag{9}$$

This is exactly the N=2 supersymmetric Euclidean action, that is constructed by Zumino [19], up to the topological term $\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}$ and the auxiliary term $\frac{1}{2}\vec{D}.\vec{D}$.

Here, we should remark that, the action belongs to the trivial cohomology of $\bar{s}$ and therefore to that of the complete operator s, if and only if the functional space where s is defined is the polynomials of the fields that are not necessarily analytic in the constant ghosts.

3 TYM as a variable redefinition

After twisting physical nature of some fields are interpreted differently, i.e. some fields become ghosts while some others become anti-ghosts [1]. In order to derive the topological fields that have the correct dimensions and ghost numbers via field redefinitions the aforementioned non-analyticity argument can be used. Since the SUSY ghosts ξ_i have ghost

[4] In other words, we assume that the action can be chosen to be a trivial element of equivariant cohomology of $\bar{s}$. See for instance Ref.s[6] and the references therein.

number one and dimension 1/2, by studying the structure of the gauge fermion Ψ_E as given in (8), the only consistent field redefinitions that assign the correct dimensionality and ghost number to the topological fields[8] are found to be

$$A_\mu = A_\mu \quad , \quad \psi_\mu = -\xi_i e_\mu \bar{\lambda}^i \quad , \quad \Phi = i\sqrt{2}\xi_i\xi^i N \quad , \quad \bar{\Phi} = \frac{i}{\sqrt{2}\xi_i\xi^i} M \tag{10}$$

$$\eta = \frac{1}{\xi_k\xi^k}\xi_i\lambda^i \quad , \quad \mathcal{X}_{\mu\nu} = \frac{-2}{\xi_k\xi^k}\xi^i e_{\mu\nu}\lambda_i \quad , \quad B_{\mu\nu} = \frac{-2}{\xi_k\xi^k}\xi^i e_{\mu\nu}\vec{\tau}_i^{\,j}\xi_j.\vec{D} \tag{11}$$

It is straightforward to show that when the above variable redefinitions are inserted in the transformations[5] (5),

$$\bar{s}A_\mu = D_\mu c + \Psi_\mu \quad , \quad \bar{s}\psi_\mu = -\{c, \Psi_\mu\} - D_\mu\Phi \quad , \quad \bar{s}\Phi = -[c, \Phi] \quad , \quad \bar{s}c = -\frac{1}{2}\{c, c\} + \Phi$$

$$\bar{s}\bar{\Phi} = -[c, \bar{\Phi}] + \eta \quad , \quad \bar{s}\eta = -\{c, \eta\} + [\Phi, \bar{\Phi}] \quad , \quad \bar{s}\mathcal{X}_{\mu\nu} = -[c, \mathcal{X}_{\mu\nu}] + F^+_{\mu\nu} + \mathcal{B}_{\mu\nu}$$

$$\bar{s}B_{\mu\nu} = -[c, B_{\mu\nu}] + [\Phi, \mathcal{X}_{\mu\nu}] - (D_\mu\psi_\nu - D_\nu\psi_\mu)^+ \tag{12}$$

one can exactly extract the scalar supersymmetry transformations δ introduced by Witten [1] if one decomposes $\bar{s}$ on the fields (A_μ, Φ, $\bar{\Phi}$, ψ_μ, η, $\mathcal{X}_{\mu\nu}$) as $\bar{s} = s_o + \delta$ [6].

Similarly, the corresponding action that can also be found by these field redefinitions

$$\begin{aligned} I_{top} &= \bar{s}\Psi_{top} = \bar{s}Tr\int d^4x(-\frac{1}{2}\eta[\Phi, \bar{\Phi}] + \frac{1}{8}\mathcal{X}_{\mu\nu}F^+_{\mu\nu} - \frac{1}{8}\mathcal{X}_{\mu\nu}B_{\mu\nu} + \bar{\Phi}D_\mu\psi_\mu) \qquad (13) \\ &= Tr\int d^4x(\frac{1}{8}F^+_{\mu\nu}F^+_{\mu\nu} + \eta D_\mu\psi_\mu - \frac{1}{4}\mathcal{X}_{\mu\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ - \bar{\Phi}D^2\Phi - \\ &\quad - \frac{1}{2}\Phi\{\eta, \eta\} - \frac{1}{8}\Phi\{\mathcal{X}_{\mu\nu}, \mathcal{X}_{\mu\nu}\} + \bar{\Phi}\{\psi_\mu, \psi_\mu\} - \frac{1}{2}[\Phi, \bar{\Phi}]^2 - \frac{1}{8}B_{\mu\nu}B_{\mu\nu}). \qquad (14) \end{aligned}$$

is exactly the Topological Yang Mills action [1] with an auxiliary field term . We remark that the inclusion of the auxiliary field is crucial in order to write the action as an exact term[7].

In other words, TYM theory in flat Euclidean space can be obtained directly as variable redefinitions from the ordinary N=2 SYM theory [8]. As it is obvious from the above definitions of the topological fields, the ghost numbers and the dimensions that are assigned to the fields in the twisting procedure by hand, appears here naturally due to the composite structure of the topological fields in terms of global ghosts ξ_i and the original fields i.e. with respect to the power of ξ_i's in the definitions.

[5] Here, $F^+_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}$ is the self-dual part of the field strength $F_{\mu\nu}$.

[6] Note that this scalar SUSY generator can also be written as a composite generator, $\delta = -i\xi^i Q_i$ where Q_i are the chiral SUSY generators.

[7] The reason why the action could not be written as an exact term in the original paper [1] is that the twisted theory was obtained from the on-shell SYM. Note that, since Ψ_{top} is gauge invariant, we have $I_{top} = \bar{s}\Psi_{top} = \delta\Psi_{top}$.

4 Baulieu-Singer formulation of TYM

Aiming to incorporate the instantons into supersymmetric theories Zumino have constructed a supersymmetric field theory directly in the Euclidean space [19]. It is then observed by Zumino that when one imposes for instance an anti self-dual field strength, i.e. $F^+_{\mu\nu} = 0$ with the restrictions $M = \lambda_i = 0$ the equations of motion from (1) reduce to a simple form [19],

$$F^+_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} = 0\,,\; D^2N = \frac{i\sqrt{2}}{2}\{\bar{\lambda}^i, \bar{\lambda}_i\}\,,\; e_\mu D_\mu \bar{\lambda}^i = 0. \tag{15}$$

that are invariant under the corresponding truncated SUSY.

The equations (15) are also the saddle point equations in the context of constraint instanton method [3]. On the other hand, similar equations are obtained in Baulieu-Singer formulation of TYM without any approximation [4]. Since, both of the approaches to the instanton calculations give the same result [4] and Wittens TYM [1] can be obtained by using simple field redefinitions (10,11), it is natural to look for another analogy between the above instanton inspired truncation of Euclidean N=2 SYM theory and the Baulieu-Singer approach to TYM.

Indeed, when the above instanton inspired truncation is used to define another nilpotent operator $\tilde{s}$,

$$\tilde{s} = \bar{s}|_{F^+_{\mu\nu} = \not{D}\bar{\lambda}^i = M = \lambda_i = 0}\;,\; \tilde{s}^2 = 0 \tag{16}$$

such that

$$\tilde{s}A_\mu = D_\mu c - \xi_i e_\mu \bar{\lambda}^i\,,\; \tilde{s}\bar{\lambda}^i = -\{c, \bar{\lambda}^i\} + i\sqrt{2}\bar{e}_\mu \xi^i D_\mu N \tag{17}$$

$$\tilde{s}N = -[c, N]\,,\; \tilde{s}c = -\frac{1}{2}\{c, c\} + i\sqrt{2}\xi_i\xi^i N \tag{18}$$

and[8]

$$\tilde{s}M = i\sqrt{2}\xi^i\lambda_i\,,\; \tilde{s}\lambda_i = \vec{\tau}_i^{\,j}\xi_j.\vec{D}\,,\; \tilde{s}\vec{D} = 0 \tag{19}$$

$\tilde{s}$-transformations are found to be exactly that of Baulieu-Singer [2] after performing the field redefinition given in (10,11) [8].

On the other hand the gauge fermion that is compatible with the restrictions of Zumino [19] has to be chosen slightly different then the one given for Euclidean case (8),

$$\Psi_{inst.} = \frac{1}{\xi_k\xi^k}Tr\int d^4x(-\frac{\alpha}{2}\xi^i\vec{\tau}_i^{\,j}\lambda_j.\vec{D} - \frac{1}{2}\xi^i e_{\mu\nu}\lambda_i F^+_{\mu\nu} + \frac{i\sqrt{2}}{2}\xi^i e_\mu \bar{\lambda}_i D_\mu M) \tag{20}$$

so that the corresponding action is

$$\begin{aligned} I^{(\alpha)}_{inst.} &= \tilde{s}\Psi_{inst.} \qquad (21)\\ &= Tr\int d^4x(-\frac{\alpha}{8}B_{\mu\nu}B_{\mu\nu} + \frac{1}{4}B_{\mu\nu}F^+_{\mu\nu} - \lambda^i e_\mu D_\mu\bar{\lambda}_i + M(D_\mu D_\mu N - \frac{i\sqrt{2}}{2}\{\bar{\lambda}^i, \bar{\lambda}_i\}) + \\ &+ \frac{1}{\xi_k\xi^k}(-\frac{1}{2}\xi^i e_{\mu\nu}\lambda_i[c, F^+_{\mu\nu}] + \frac{i\sqrt{2}}{2}M\{c, \xi^i e_\mu D_\mu\bar{\lambda}_i\})\;) + \frac{1}{\xi_k\xi^k}Tr\int d^4x\partial_\mu(\tilde{s}\frac{i\sqrt{2}}{2}M\xi^i e_\mu\bar{\lambda}_i) \end{aligned}$$

[8] The reason why we do not set $\lambda_i = \vec{D} = 0$ in Eq.(20) is that the pairs $(M, \xi^i\lambda_i)$ and $(\xi^i\vec{\tau}_i^{\,j}\lambda_j, \vec{D})$ behaves like the trivial pairs (BRST doublets) It is known that the cohomology of an operator does not depend on inclusion of such trivial pairs (see for instance [10, 17]).

where we have used the definition of $B_{\mu\nu}$ in order to have notational simplification.

First of all, the gauge fermion Ψ_{inst} (20) and the above action I_{inst} are exactly the ones given in Baulieu-Singer approach [2] up to ordinary gauge fixing. However, if the above relations are considered on their own, to be able to derive the instanton equations (15) from the action functional without having any dependence on the constant ghosts, the coefficient of $Tr\xi^i\lambda_i[M,N]$ in the Euclidean Ψ_E has to be chosen to vanish whereas the coefficient α of $Tr\tilde{s}\xi^i\vec{\tau}_i^j\lambda_j.\vec{D}$ can be left arbitrary. Therefore, the gauge fermion Ψ_{inst} is the only consistent choice up to total derivatives that gives the right action to derive the exact instanton equations, when the truncated transformations (17-19) are used.

Acknowledgments: I am grateful to the organizers of the Workshop SQS'05 for inviting me and for the kind hospitality during the workshop.

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